Number of Submatrices of a Matrix

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Problem

How many submatrices of a matrix of dimension $m \times n$?

Intuition

Number of substrings of non-zero length of a string of length $m$ is $\frac{m(m+1)}{2}$. Also, an $m \times 1$ matrix can be viewed a string of length $m$ and the submatrices of that matrix are the substrings of the given string, and hence we expect the formula for number of submatrices to reduce to $\frac{m(m+1)}{2}$, when $n = 1$. Also, intuitively, we can guess that the number of submatrices should at least be proportional of square of $m$ and $n$, i.e. $f(m, n) = \Omega(m^2n^2)$, where $f(m, n)$ is the value which we are seeking, i.e. number of submatrices of an $m \times n$ matrix.

Also, observe that number of submatrices of an $m \times n$ matrix is same as that of a $n \times m$ matrix. Hence, $f(m, n) = f(n, m)$, i.e. we expect the closed form of the formula to be symmetric in $m$ and $n$.

A Recursive Formula

Let $f(m, n)$ be the number of submatrices of a matrix with $m$ rows and $n$ columns.

Since there is only one submatrix of a $1 \times 1$ matrix, $f(1, 1) = 1$.

Also, as we noted earlier, $f(m, 1) = f(1, m) = \frac{m(m+1)}{2}$.

Now, assume that one more column is added to the currently existing $m \times n$ matrix. How many new submatrices are added to $f(m, n)$?

There are $\frac{m(m+1)}{2}$ submatrices of width 1 (this is analogous to substrings). For each such matrix, in corresponding rows, there are $n + 1$ submatrices (exactly one of width $1, 2, 3, \ldots, n + 1$). Hence, in total there are $\frac{m(m+1)(n+1)}{2}$ submatrices which are newly added. This can be written as a recursive formula:

$$f(m, n + 1) = f(m, n) + \frac{m(m+1)(n+1)}{2}.$$
Closed Form

\[ f(m, n) = f(m, n - 1) + \frac{m(m + 1)n}{2} \]
\[ = f(m, n - 2) + \frac{m(m + 1)(n - 1)}{2} + \frac{m(m + 1)n}{2} \]
\[ = \ldots \]
\[ = f(m, 1) + \frac{m(m + 1)2}{2} + \ldots + \frac{m(m + 1)(n - 1)}{2} + \frac{m(m + 1)n}{2} \]
\[ = \frac{m(m + 1)}{2} + \frac{m(m + 1)2}{2} + \ldots + \frac{m(m + 1)(n - 1)}{2} + \frac{m(m + 1)n}{2} \]
\[ = \frac{m(m + 1)}{2}(1 + 2 + \ldots + n) \]
\[ = \frac{m(m + 1)n(n + 1)}{2} \]
\[ = \frac{m(m + 1)n(n + 1)}{2} \]
\[ = \frac{m(m + 1)n(n + 1)}{4} \]

Replacing \( n + 1 \) by \( n \).
Applying same formula again
\[ f(m, 1) = \frac{m(m + 1)}{2} \].
\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]

This closed form of \( f(m, n) \) is indeed symmetric in \( m \) and \( n \).
Also, putting \( n = 1 \), it reduces to \( f(m, 1) = \frac{m(m + 1)}{2} \).
And as we hypothesized, \( f(m, n) = \Omega(m^2n^2) \), in fact \( f(m, n) = \theta(m^2n^2) \)!

A Simpler Derivation

The earlier derivation was a recursive one, and it took quite some steps to get a closed form. Now, observing the closed form, we can think of an extremely simple way to get the same formula. An \( m \times n \) matrix can be thought of as \( m + 1 \) horizontal and \( n + 1 \) vertical lines. Hence, process of forming a submatrix can be broken into 2 independent steps:

1. Choose 2 (upper and lower) boundaries from \( m + 1 \) horizontal lines: \( \binom{m+1}{2} \) ways.
2. Choose 2 (left and right) boundaries from \( n + 1 \) vertical lines: \( \binom{n+1}{2} \) ways.

Hence, by product rule, total number of submatrices = \( \binom{m+1}{2} \cdot \binom{n+1}{2} = \frac{m(m + 1)n(n + 1)}{4} \).

Yet Another Derivation!

Now, let us calculate the number of submatrices by summing up all submatrices of all possible sizes.
It is not difficult to see that number of submatrices of size \( a \times b \) in an \( m \times n \) matrix is \( (m - (a + 1)) \cdot (n - (b + 1)) \).
This can be derived from using ideas from the previous derivation.
An \( a \times b \) matrix, by definition, occupies \( a \) rows and \( b \) columns. A submatrix having \( a \) rows can have its upper boundary at any of the first \( m - a + 1 \) horizontal lines of the original matrix (total horizontal lines, as stated before, are \( m + 1 \)). By similar argument, it can have its left boundary at \( n - b + 1 \) vertical lines.
Hence, total number of \( a \times b \) matrices, by product rule, is \( (m - a + 1) \times (n - b + 1) \).
Now, total number of submatrices can be found by summing this over all \( a \)s and \( b \)s.
Hence,

\[ f(m, n) = \sum_{a=1}^{m} \sum_{b=1}^{n} (m - a + 1) \cdot (n - b + 1) \]

\[ = \sum_{a=1}^{m} \sum_{b=1}^{n} (mn - mb + m - an + ab + n - b + 1) \]

Expanding

\[ = \sum_{a=1}^{m} \sum_{b=1}^{n} (mn + m + n + 1) + \sum_{a=1}^{m} \sum_{b=1}^{n} b(m + 1) + \sum_{a=1}^{m} \sum_{b=1}^{n} a(n + 1) - \sum_{a=1}^{m} \sum_{b=1}^{n} (ab) \]

Collecting like terms

\[ = (mn + m + n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} 1 - (m + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} b - (n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} a + \sum_{a=1}^{m} \sum_{b=1}^{n} ab \]

Changing order

\[ = (m + 1)(n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} 1 - (m + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} b - (n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} a + \sum_{a=1}^{m} \sum_{b=1}^{n} ab \]

\[ = (m + 1)(n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} 1 - (m + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} b - (n + 1) \sum_{a=1}^{m} \sum_{b=1}^{n} a + \sum_{a=1}^{m} \sum_{b=1}^{n} ab \]

\[ = (m + 1)(n + 1) mn - \frac{(m + 1)(n + 1)nm}{2} - \frac{(n + 1)m(m + 1)n}{2} + \frac{m(m + 1)n(n + 1)}{2} \]

\[ = m(m + 1)n(n + 1) - \frac{2m(m + 1)n(n + 1)}{2} + \frac{m(m + 1)n(n + 1)}{4} \]

\[ = \frac{m(m + 1)n(n + 1)}{4} \]